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The generalized transmuted-G family of distributions

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ABSTRACT
We introduce a new class of continuous distributions called the generalized transmuted-G family which extends the transmuted-G class. We provide six special models of the new family. Some of its mathematical properties including explicit expressions for the ordinary and incomplete moments, generating function, Rényi and Shannon entropies, order statistics and probability weighted moments are derived. The estimation of the model parameters is performed by maximum likelihood. The flexibility of the proposed family is illustrated by means of three applications to real data sets.

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1. Introduction

Several continuous univariate distributions have been extensively used in the literature for modeling data in many areas such as engineering, economics, biological studies and environmental sciences. However, applied areas such as lifetime analysis, finance, and insurance clearly require extended forms of these distributions. Therefore, several classes of distributions have been constructed by extending common families of continuous distributions. These generalized distributions give more flexibility by adding one or more parameters to the baseline model. Gupta et al. (1998) pioneered the exponentiated-G (E-G) class, which consists of raising the cumulative distribution function (cdf) to a positive power parameter. Many other classes can be mentioned such as the beta generalized (BG) family by Eugene et al. (2002), Kumaraswamy (Kw-G) family by Cordeiro and de Castro (2011), and exponentiated generalized (EG) family by Cordeiro et al. (2013).

Consider a baseline cdf $G(x; \varphi)$ and probability density function (pdf) $g(x; \varphi)$ depending on a parameter vector φ, where $\varphi = (\varphi_k) = (\varphi_1, \varphi_2, \ldots)$. Thus, the cdf and pdf of the *transmuted class* (TC) of distributions are defined by

$$F(x; \lambda, \varphi) = G(x; \varphi)\left[(1+\lambda) - \lambda G(x; \varphi)\right] \tag{1}$$

and

$$f(x; \lambda, \varphi) = g(x; \varphi)\ \left[1 + \lambda - 2\lambda G(x; \varphi)\right], \tag{2}$$

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respectively. The TC density is a mixture of the baseline density and the E-G density with power parameter two. Furthermore, $\lambda = 0$ in(2) gives the baseline distribution. Further details were explored by Shaw and Buckley (2007).

In this paper, we propose the *generalized transmuted-G* (GT-G) family of distributions, which extends the TC model by incorporating two additional shape parameters to generate more flexible distributions. The main advantage of the new family relies on the fact that practitioners will have a quite flexible three-parameter generator to fit real data from several fields. It may serve as a good alternative to other two and three-parameter classes. We also hope that the proposed family may work better (at least in terms of model fitting) than other classes of distributions in certain practical situations, although it cannot always be guaranteed. Additionally, we provide a comprehensive account of some of its mathematical properties. As we will see later, the formulae related to the proposed family are simple and manageable, and with the use of modern computer resources and their numerical capabilities, it may prove to be a useful addition to those distributions applied for modelling data in economics, medicine, reliability and engineering, among other areas.

The rest of the paper is organized as follows. In Section 2, we define the GT-G family and give a very useful representation for its density function. In Section 3, we present six special models corresponding to the baseline Weibull, Lomax, Burr X, log-logistic, Lindley and Weibull geometric distributions and plots of their pdfs. In Section 4, we derive some of its mathematical properties including ordinary and incomplete moments, mean deviations, moment generating function (mgf), Rényi, Shannon and q-entropies. Order statistics and their moments are investigated in Section 5. In Section 6, we obtain the probability weighted moments (PWMs) of the new family. Maximum likelihood estimation of the model parameters is addressed in Section 7. In Section 8, we provide three applications to real data to illustrate the flexibility of the new family. Finally, some concluding remarks are addressed in Section 9.

2. The GT-G family

The cdf of the GT-G family is defined by

$$F(x) = F(x; \lambda, a, b, \varphi) = G(x; \varphi)^a \left[(1+\lambda) - \lambda G(x; \varphi)^b\right]. \tag{3}$$

The pdf corresponding to (3) is given by

$$f(x; \lambda, a, b, \varphi) = g(x; \varphi)\, G(x; \varphi)^{a-1} \left[a(1+\lambda) - \lambda(a+b)\, G(x; \varphi)^b\right]. \tag{4}$$

We denote by $X \sim$GT-G$(\lambda, a, b, \boldsymbol{\varphi})$ a random variable having density function (4). Further, we will omit the dependence on the model parameters and write simply $g(x) = g(x; \boldsymbol{\varphi})$, $F(x) = F(x; \lambda, a, b, \boldsymbol{\varphi})$ and $f(x) = f(x; \lambda, a, b, \boldsymbol{\varphi})$, etc. Some special cases of the new family are listed in Table 1.

Table 1. Sub-models of the GT-G family.

a	b	λ	$G(x; \varphi)$	Reduced Model	Author
1	1	λ	$G(x; \varphi)$	T-G family	Shaw and Buckley (2007)
a	0	—	$G(x; \varphi)$	E-G family	Gupta et al. (1998)
1	1	0	$G(x; \varphi)$	$G(x; \varphi)$	—

2.1. Mixture representation

We provide a useful representation for (4) using the concept of exponentiated class. Henceforth, for an arbitrary baseline cdf $G(x;\boldsymbol{\varphi})$, let Y_δ be a random variable following the E-G class with power parameter $\delta > 0$, say $Y_\delta \sim$E-G$(\delta,\boldsymbol{\varphi})$, i.e., its cdf and pdf are $H_\delta(x) = G(x;\boldsymbol{\varphi})^\delta$ and $h_\delta(x) = \delta\, g(x;\boldsymbol{\varphi})\, G(x;\boldsymbol{\varphi})^{\delta-1}$, respectively. The properties of the exponentiated distributions have been studied by many authors in the last twenty years.

Therefore, the GT-G family density in (4) can be expressed as

$$f(x) = (1+\lambda)\, h_a\,(x;\,\varphi) - \lambda\, h_{a+b}\,(x;\,\varphi)\,. \tag{5}$$

Equation (5) reveals that the GT-G density function is a mixture of two E-G densities. Thus, some mathematical properties of the new family can be derived from those properties of the E-G class. For example, the ordinary and incomplete moments and mgf of X can be obtained directly from those quantities of the E-G class.

3. Special models

In this section, we provide six special cases of the GT-G family. The pdf (4) will be most tractable when $G\,(x;\,\varphi)$ and $g\,(x;\,\varphi)$ have simple analytic expressions. These special models generalize some well-known distributions in the literature. Here, we provide six special models of this family corresponding to the baseline Weibull (W), Lomax (Lo), Burr X (BrX), log-logistic (LL), Lindley (Li) and Weibull geometric (WG) distributions. The pdf and cdf (for $x > 0$) of these baseline models are listed in Table 2.

The parameters of the above densities are all positive real numbers except for the WG distribution, where $p \in (0,\,1)$.

3.1. The GT-W distribution

The GT-W pdf is given by

$$f(x) = \beta\alpha^\beta x^{\beta-1} \exp\left[-(\alpha x)^\beta\right]\left\{1-\exp\left[-(\alpha x)^\beta\right]\right\}^{a-1}$$
$$\times\left\{a\,(1+\lambda) - \lambda\,(a+b)\left(1-\exp\left[-(\alpha x)^\beta\right]\right)^b\right\}.$$

The GT-W distribution includes the transmuted Weibull (TW) distribution when $a = b = 1$. For $b = 0$, we obtain the exponentiated Weibull (EW) distribution. For $\beta = 2$, we obtain

Table 2. The pdfs and cdfs of baseline models.

Model	$g(x)$	$G(x)$
W	$\beta\alpha^\beta x^{\beta-1}\, \mathrm{e}^{-(\alpha x)^\beta}$	$1-\mathrm{e}^{-(\alpha x)^\beta}$
Lo	$(\alpha/\beta)\left[1+(x/\beta)\right]^{-(\alpha+1)}$	$1-\left[1+(x/\beta)\right]^{-\alpha}$
BrX	$2\alpha\beta^2\, x\, \mathrm{e}^{-(\beta x)^2}\left\{1-\mathrm{e}^{-(\beta x)^2}\right\}^{\alpha-1}$	$\left\{1-\mathrm{e}^{-(\beta x)^2}\right\}^\alpha$
LL	$\beta\alpha^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^\beta\right]^{-2}$	$1-\left[1+\left(\frac{x}{\alpha}\right)^\beta\right]^{-1}$
Li	$\frac{\alpha^2}{1+\alpha}(1+x)\mathrm{e}^{-\alpha x}$	$1-\frac{1+\alpha+\alpha x}{1+\alpha}\mathrm{e}^{-\alpha x}$
WG	$\alpha\beta^\alpha\,(1-p)\,x^{\alpha-1}\mathrm{e}^{-(\beta x)^\alpha}\left\{1-p\mathrm{e}^{-(\beta x)^\alpha}\right\}^{-2}$	$\frac{1-\mathrm{e}^{-(\beta x)^\alpha}}{1-p\mathrm{e}^{-(\beta x)^\alpha}}$

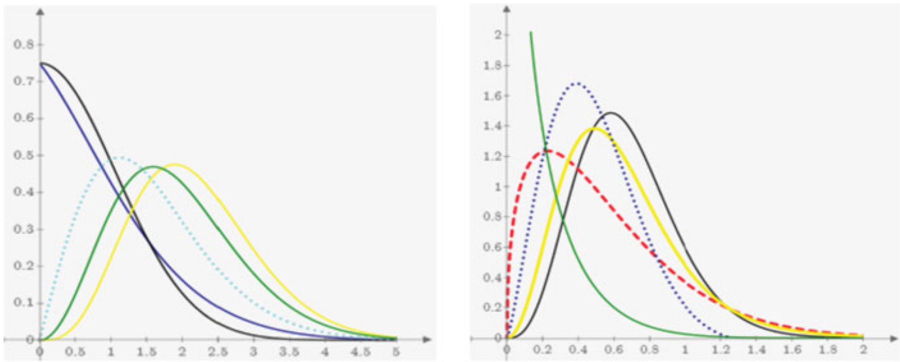


Figure 1. The GT-W pdf: (a) For $\alpha = \lambda = .5$ and $\beta = 2$: $a = b = .5$ (blue line), $a = .5$ and $b = 1$ (black line), $a = 2$ and $b = .8$ (yellow line), $a = 1$, and $b = .9$ (dotted line) and $a = 1.5$ and $b = .8$ (green line) (b) For $\alpha = \beta = 1.5$ and $\lambda = 1$: $a = b = 2.5$ (black line), $a = 1$ and $b = .2$ (dashed line), $a = 2$ and $b = 1.5$ (yellow line) $a = 1.5$, and $b = 2$ (dotted line) and $a = b = .4$ (green line).

the GT-Rayleigh (GT-R) distribution. For $\beta = 1$, we have the GT-exponential (GT-E) distribution. Plots of the GT-W density for some parameter values are displayed in Figure 1.

3.2. The GT-Lo distribution

The GT-Lo density is given by

$$f(x) = \left(\frac{\alpha}{\beta}\right)\left[1 + \left(\frac{x}{\beta}\right)\right]^{-(\alpha+1)} \left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{a-1} \times \left\{a(1+\lambda) - \lambda(a+b)\left\{1 - \left[1 + \left(\frac{x}{\beta}\right)\right]^{-\alpha}\right\}^{b}\right\}.$$

The GT-Lo distribution includes the transmuted Lomax (TLo) distribution when $a = b = 1$. For $b = 0$, we obtain the exponentiated Lomax (ELo) distribution. The plots of the GT-Lo density are displayed in Figure 2 for some parameter values.

3.3. The GT-BrX distribution

The GT-BrX density is given by

$$f(x) = 2\alpha\beta^2 x \exp\left[-(\beta x)^2\right]\left\{1 - \exp\left[-(\beta x)^2\right]\right\}^{\alpha a-1} \times \left\{a(1+\lambda) - \lambda(a+b)\left(1 - \exp\left[-(\beta x)^2\right]\right)^{\alpha b}\right\}.$$

The GT-BrX distribution includes the transmuted Burr X (T-BrX) distribution when $a = b = 1$. For $b = 0$, we obtain the exponentiated Burr X (EBrX) distribution. We display some possible shapes of the GT-BrX density function in Figure 3.

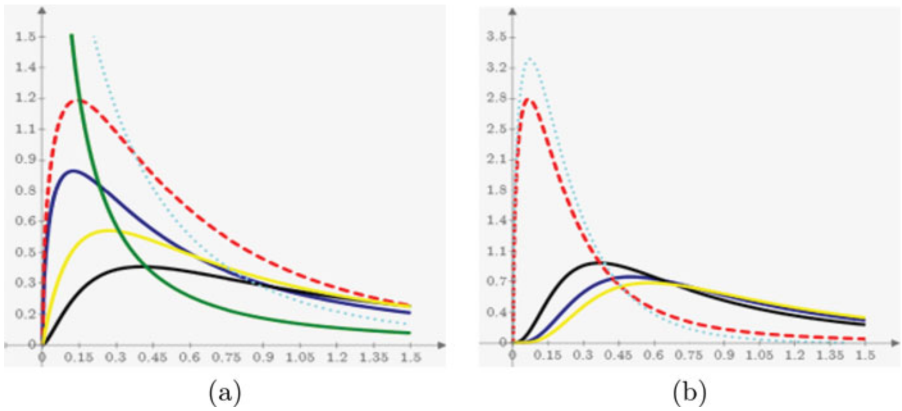


Figure 2. The GT-Lo pdf: (a) For $\alpha = .75$ and $\beta = .5$: $\lambda = .5$, $a = 1.5$ and $b = 1.3$ (blue line), $\lambda = .5$, $a = 2.5$ and $b = 1.5$(black line), $\lambda = 1$, $a = 1.5$ and $b = 3$ (dashed line), $\lambda = .5$ and $a = b = 2$ (yellow line), $\lambda = a = 1$ and $b = 2$ (dotted line) and $\lambda = a = b = .5$ (green line) (b) For $\alpha = 2$, $\beta = .5$ and $a = 1.5$: $\lambda = -1$ and $b = 4$ (blue line), $\lambda = -1$ and $b = 2.5$(black line), $\lambda = .5$ and $b = 2.5$(dashed line), $\lambda = -1$ and $b = 5$ (yellow line) and $\lambda = .7$ and $b = 3$(dotted line).

3.4. The GT-LL distribution

The GT-LL density is given by

$$f(x) = \beta\alpha^{-\beta}x^{\beta-1}\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-2}\left\{1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right\}^{a-1}$$
$$\times\left\{a\,(1+\lambda)-\lambda\,(a+b)\left(1-\left[1+\left(\frac{x}{\alpha}\right)^{\beta}\right]^{-1}\right)^{b}\right\}.$$

The GT-LL distribution includes the transmuted log-logistic (TLL) model when $a = b = 1$. For $b = 0$, we obtain the exponentiated log-logistic (ELL) distribution. Figure 4 gives plots of the GT-LL density for selected parameter values.

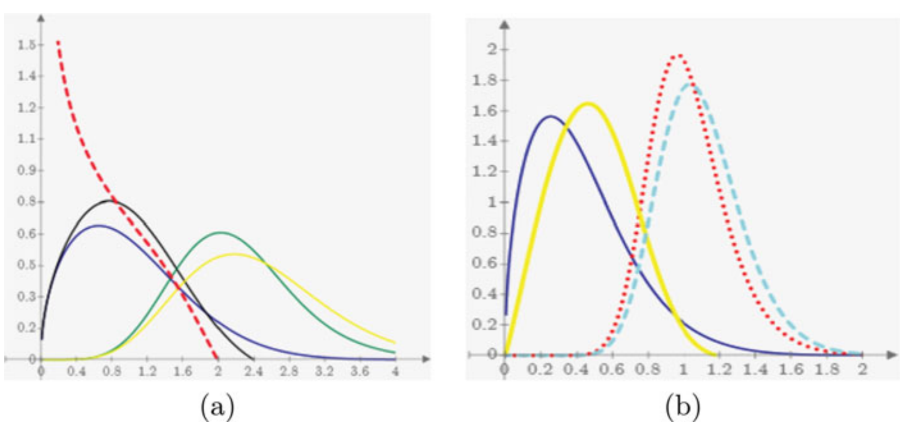
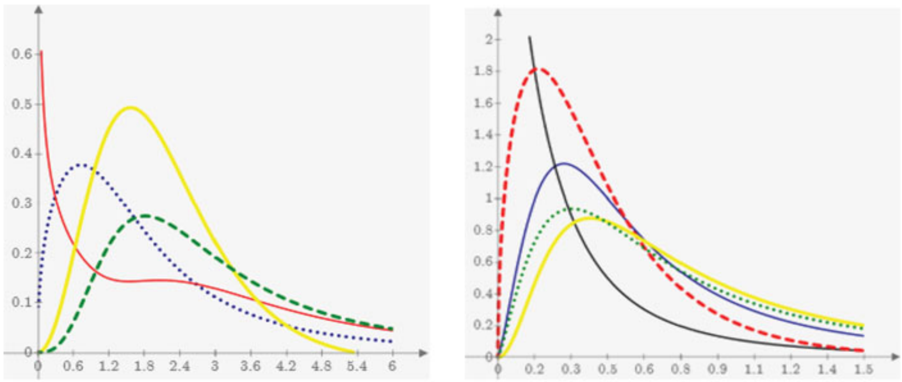


Figure 3. The GT-BrX pdf: (a) For $\alpha = 1.5$, $\beta = .5$ and $\lambda = 1$: $a = b = .5$ (blue line), $a = .5$ and $b = 1$ (black line), $a = 2$ and $b = 1.5$ (green line), $a = 2$, and $b = .5$ (yellow line) and $a = .2$ and $b = 3$ (dashed line) (b) For $\alpha = \beta = 1.5$: $\lambda = a = b = .5$ (blue line), $\lambda = .7$, $a = 6$ and $b = 4$(dotted line), $\lambda = .3$, $a = .7$ and $b = 3$ (yellow line) and $\lambda = -1$, $a = 4$ and $b = 2$ (dashed line).



(a) (b)

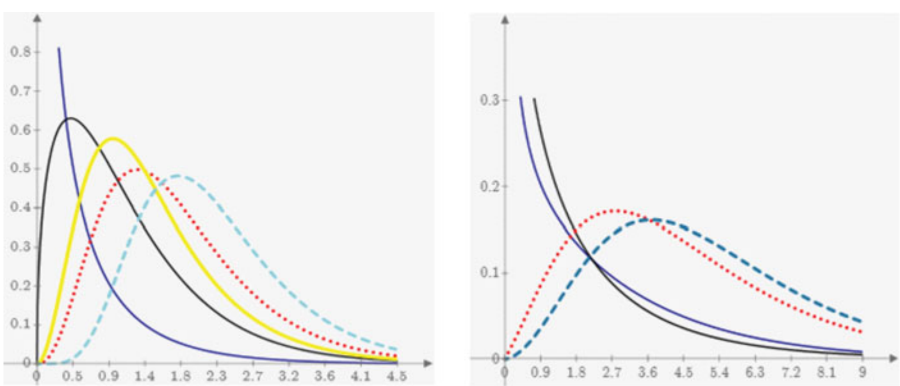
Figure 4. The GT-LL pdf: (a) For $\alpha = \beta = 2$: $\lambda = -1$, $a = .2$ and $b = .5$ (dotted line), $\lambda = -.5$, $a = .3$ and $b = 3$(red line), $\lambda = .8$, $a = 1.5$ and $b = 5$(yellow line) and $\lambda = .7$, $a = 2$ and $b = .5$(dashed line) (b) For $\alpha = \lambda = .5$ and $\beta = 1.5$: $a = b = 1.5$(blue line), $a = b = .5$(black line), $a = 1$ and $b = 3$(dashed line) $a = 2$ and $b = .8$ (yellow line) and $a = 1.5$ and $b = .1$(dotted line).

3.5. The GT-Li distribution

The GT-Li pdf is given by

$$f(x) = \frac{\alpha^2}{1+\alpha}(1+x)\exp(-\alpha x)\left[1 - \frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]^{a-1} \times \left\{a(1+\lambda) - \lambda(a+b)\left[1 - \frac{1+\alpha+\alpha x}{1+\alpha}\exp(-\alpha x)\right]^{b}\right\}.$$

The GT-Li distribution includes the transmuted Lindley (TLi) model when $a = b = 1$. For $b = 0$, we obtain the exponentiated Lindley (ELi) distribution. Plots of the GT-Li density function are displayed in Figure 5 for some parameter values.



(a) (b)

Figure 5. The GT-Li pdf: (a) For $\alpha = 1.5$: $\lambda = a = b = .5$ (blue line), $\lambda = .3$, $a = 1.5$ and $b = .5$(black line), $\lambda = -1$, $a = 2$ and $b = 1.5$(dotted line), $\lambda = .5$, $a = 3$ and $b = 2$ (yellow line) and $\lambda = -1$, $a = 5$ and $b = 1.5$(dashed line) (b) For $\alpha = b = .5$: $\lambda = -.5$ and $a = .2$(blue line), $\lambda = a = .5$(black line), $\lambda = .5$ and $a = 2$(dotted line) and $\lambda = -1$ and $a = 2$(dashed line).

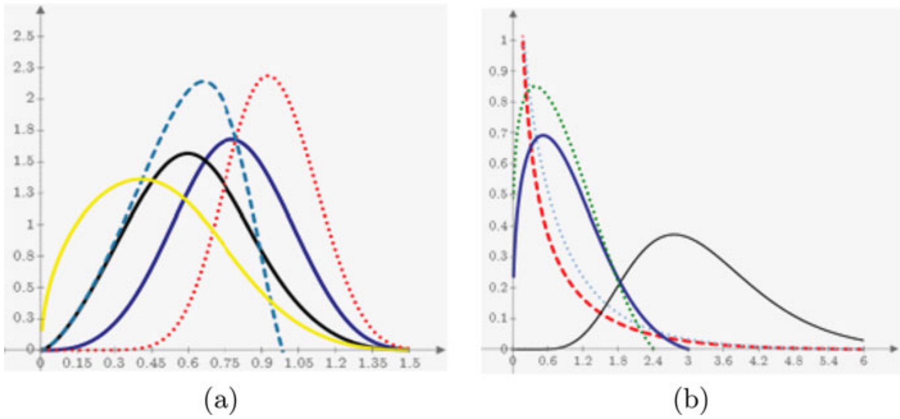


Figure 6. The GT-WG pdf: (a) For $\alpha = 1$, $\beta = 5$ and $\lambda = p = .5$: $a = .8$ and $b = .1$ (blue line), $a = b = \lambda = .5$ (black line), $a = 2$ and $b = .5$ (dotted line), $a = .5$, and $b = 3$ (dashed line) and $a = b = .3$ (yellow line) (b) For $\alpha = .5$, $\beta = 1.5$ and $p = .3$: $a = b = \lambda = .3$ (dashed line), $\lambda = -1$ and $a = b = 3$ (black line), $a = b = \lambda = .5$(dotted line) $\lambda = .5$, $a = .9$ and $b = 3$ (blue line) and $\lambda = .7$, $a = .8$ and $b = 4$ (dotted-green line).

3.6. The GT-WG distribution

The GT-WG density is given by

$$f(x) = \alpha\beta^{\alpha}\left(1-p\right)x^{\alpha-1}\left\{1-p\exp\left[-(\beta x)^{\alpha}\right]\right\}^{-2}$$
$$\times \exp\left(-(\beta x)\right)^{\alpha}\left\{\frac{1-\exp(-(\beta x)^{\alpha}}{1-p\exp\left[-(\beta x)^{\alpha}\right]}\right\}^{a-1}$$
$$\times\left\{a\left(1+\lambda\right)-\lambda\left(a+b\right)\left(\frac{1-\exp\left[-(\beta x)^{\alpha}\right]}{1-p\exp\left[-(\beta x)^{\alpha}\right]}\right)^{b}\right\}.$$

The GT-WG distribution includes the transmuted Weibull Geometric (TWG) model when $a = b = 1$. For $b = 0$, we obtain the exponentiated Weibull Geometric (EWG) distribution. Figure 6 gives possible shapes of the GT-WG density function for some parameter values.

4. Mathematical properties

The formulae derived throughout the paper can be easily handled in most symbolic computation software platforms such as **Maple**, **Mathematica** and **Matlab**. These platforms have currently the ability to deal with analytic expressions of formidable size and complexity. Established explicit expressions to calculate statistical measures can be more efficient than computing them directly by numerical integration.

4.1. Moments

The nth ordinary moment of X, say μ'_n, follows from (5) as

$$\mu'_n = E(X^n) = (1+\lambda)E(Y_a^n) - \lambda E(Y_{a+b}^n). \tag{6}$$

For $\delta > 0$, we have $E(Y_\delta^n) = \delta\int_{-\infty}^{\infty} x^n\, g(x;\boldsymbol{\varphi})\, G(x;\boldsymbol{\varphi})^{\delta-1}\, dx$, which can be computed numerically in terms of the baseline quantile function (qf) $Q_G(u;\boldsymbol{\varphi}) = G^{-1}(x;\boldsymbol{\varphi})$ as

$$E(Y_\delta^n) = \delta\int_0^1 Q_G(u;\varphi)^n\, u^{\delta-1}du.$$

Setting $n = 1$ in (6) gives the mean of X. The central moments (μ_s) and cumulants (κ_s) of X are determined from (6) as $\mu_s = \sum_{k=0}^{s} (-1)^k \binom{s}{k} \mu_1'^k \mu_{s-k}'$ and $\kappa_s = \mu_s' - \sum_{k=1}^{s-1} \binom{s-1}{k-1} \kappa_k \mu_{s-k}'$, respectively, where $\kappa_1 = \mu_1'$. The skewness $\gamma_1 = \kappa_3/\kappa_2^{3/2}$ and kurtosis $\gamma_2 = \kappa_4/\kappa_2^2$ are obtained from the third and fourth standardized cumulants.

4.2. Incomplete moments

The nth incomplete moment of X is defined by $m_n(y) = \int_{-\infty}^{y} x^n f(x)dx$. We have

$$m_n(y) = (1+\lambda)m_{n,a}(y) - \lambda m_{n,a+b}(y), \tag{7}$$

where

$$m_{n,\delta}(y) = \int_0^{G(y;\,\varphi)} Q_G(u;\varphi)^n\, u^{\delta-1}\, du.$$

The integral $m_{n,\delta}(y)$ can be determined analytically for special models with closed-form expressions for $Q_G(u;\boldsymbol{\varphi})$ or computed at least numerically for most baseline distributions.

An important application of the first incomplete moment of X in (7), say $m_1(y)$, refers to the Bonferroni and Lorenz curves. These curves are very useful in economics, reliability, demography, insurance and medicine. For a given probability π, the Bonferroni and Lorenz curves are given by $B(\pi) = m_1(p)/(p\mu_1')$ and $L(p) = m_1(p)/\mu_1'$, where $p = Q(\pi) = F^{-1}(\pi)$ can be determined numerically by inverting (3).

Another application refers to the mean deviations about the mean ($\delta_1 = E(|X - \mu_1'|)$) and about the median ($\delta_2 = E(|X - M|)$) of X given by

$$\delta_1 = 2\mu_1' F(\mu_1') - 2m_1(\mu_1') \qquad \text{and} \qquad \delta_2 = \mu_1' - 2m_1(M),$$

respectively, where M is the median of X, $\mu_1' = E(X)$ comes from equation (6), $F(\mu_1')$ is easily calculated from (3) and $m_1(z)$ comes from (7) with $n = 1$.

4.3. Generating function

The mgf of X, say $M(t) = E(e^{tX})$, is obtained from (5) as

$$M(t) = (1+\lambda)M_a(t;\varphi) - \lambda M_{a+b}(t;\varphi),$$

where $M_\delta(t;\boldsymbol{\varphi})$ is the generating function of Y_δ given by

$$M_\delta(t;\varphi) = \delta \int_{-\infty}^{\infty} e^{tx}\, G(x;\varphi)^{\delta-1} g(x;\varphi)dx = \delta \int_0^1 \exp\left[t\, Q_G(u;\delta)\right] u^{\delta-1} du.$$

The last two integrals can be computed numerically for most parent distributions.

4.4. Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_\theta(X) = \frac{1}{1-\theta} \log \int_{-\infty}^{\infty} f(x)^\theta\, dx, \;\; \theta > 0 \text{ and } \theta \neq 1.$$

By using the pdf in (4), we can write

$$f(x)^\theta = (1+\lambda)^\theta\, h_a(x)^\theta \left\{1 - d\, G(x)^b\right\}^\theta,$$

where $d = \lambda\,(a+b)/[a\,(1+\lambda)]$.

Then, the Rényi entropy of a random variable X having the GT-G family is given by

$$I_\theta(X) = \frac{1}{1-\theta} \log\left\{(1+\lambda)^\theta \int_{-\infty}^{\infty} h_a(x)^\theta \left\{1 - dG(x)^b\right\}^\theta dx\right\}.$$

The q-entropy, say $H_q(X)$, is given by

$$H_q(X) = \frac{1}{q-1} \log\left\{1 - \int_{-\infty}^{\infty} f(x)^q\, dx\right\}, q > 0 \text{ and } q \neq 1,$$

and then

$$H_q(X) = \frac{1}{q-1} \log\left\{1 - \left[(1+\lambda)^q \int_{-\infty}^{\infty} h_a(x)^q \left\{1 - d\,G(x)^b\right\}^q dx\right]\right\},$$

for $q > 0$ and $q \neq 1$.

The Shannon entropy, say SI, of a random variable X is defined by

$$SI = E\left\{-\left[\log f(X)\right]\right\}.$$

It is a special case of the Rényi entropy when $\theta \uparrow 1$. Therefore, based on Equation (5), we can write

$$SI = -\left\{\log\left[(1+\lambda)\, J_a - \lambda\, J_{a+b}\right]\right\},$$

where

$$J_k = k \int_0^{\infty} x\, g(x)\; G(x)^{k-1}\, dx.$$

The last equation can be determined numerically for any G model.

5. Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the GT-G family. The pdf of the ith order statistic, say $X_{i:n}$, is given by

$$f_{i:n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{j+i-1}(x). \tag{8}$$

We can write from (3)

$$F^{j+i-1}(x) = (1+\lambda)^{j+i-1}\; G(x)^{a(j+i-1)}\; \left\{1 - s\,G(x)^b\right\}^{j+i-1},$$

where $s = \lambda/(\lambda+1)$. Using the power series expansion, the last equation can be expressed as

$$F^{j+i-1}(x) = \sum_{k=0}^{\infty} w_k\, G(x)^{kb+a(j+i-1)}, \tag{9}$$

where $w_k = (-1)^k\, \Gamma(j+i) s^k\, (1+\lambda)^{j+i-1}/[k!\Gamma(j+i-k)]$.

By inserting (4) and (9) in Equation (8), the pdf of $X_{i:n}$ becomes

$$f_{i:n}(x) = \sum_{k=0}^{\infty} \sum_{j=0}^{n-i} w_k^* \left\{m\, h_{kb+a(j+i)}(x) - q\, h_{b(k+1)+a(j+i)}(x)\right\}, \tag{10}$$

where $m = m(k,j) = \frac{a(1+\lambda)}{kb+a\left(j+i\right)}$, $q = q(k,j) = \frac{\lambda(a+b)}{b(k+1)+a\left(j+i\right)}$ and

$$w_k^* = \frac{(-1)^{k+j}\,\Gamma\left(j+i\right)s^k\,(1+\lambda)^{j+i-1}}{k!\,\Gamma\left(j+i-k\right)\mathrm{B}\left(i,n-i+1\right)}\binom{n-i}{j}.$$

Thus, the density function of the GT-G order statistics is a mixture of E-G densities. Based on Equation (10), we can obtain some structural properties of $X_{i:n}$ from those of the E-G model.

The rth moment of $X_{i:n}$ is given by

$$E\left(X_{i:n}^r\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{n-i} w_k^*\left\{m\,E\left[Y_{kb+a\left(j+i\right)}^r\right] - q\,E\left[Y_{b(k+1)+a\left(j+i\right)}^r\right]\right\}, \tag{11}$$

where Y_δ (as before) has the E-G density with power parameter δ.

The L-moments are analogous to the ordinary moments but can be estimated by linear combinations of order statistics. Based upon the moments in equation (11), we can derive explicit expressions for the L-moments of X as infinite weighted linear combinations of the means of suitable E-G distributions. They are linear functions of expected order statistics defined by

$$\lambda_r = \frac{1}{r}\sum_{d=0}^{r-1}(-1)^d\binom{r-1}{d}E\left(X_{r-d:r}\right),\; r \geq 1.$$

6. Probability weighted moments

The PWMs are expectations of certain functions of a random variable and they can be defined for any random variable whose ordinary moments exist. These moments can generally be used for estimating parameters of a distribution whose inverse form cannot be expressed explicitly.

The (s,r)th PWM of X following the GT-G family, say $\rho_{s,r}$, is formally defined by

$$\rho_{s,r} = E\left\{X^s\,F(X)^r\right\} = \int_{-\infty}^{\infty} x^s\,F(x)^r\,f\left(x\right)\,dx.$$

From Equations (3) and (4), we can write

$$f\left(x\right)\,F(x)^r = \sum_{k=0}^{\infty}\upsilon_k\,h_{bk+a(r+1)}\left(x\right)\,\{1 - d\,G\left(x\right)^b\},$$

where $\upsilon_k = \frac{(-1)^k\,a\,\Gamma(r+1)\,s^k\,(1+\lambda)^{r+1}}{k!\,\Gamma(r-k+1)\,(bk+a[r+1])}$ and, as before, $h_\delta\left(x\right) = \delta g\left(x\right)\,G\left(x\right)^{\delta-1}$.

Then, $\rho_{s,r}$ can be expressed as

$$\rho_{s,r} = \sum_{k=0}^{\infty}\upsilon_k\int_0^{\infty} x^s\,h_{bk+a(r+1)}\left(x\right)\,\{1 - d\,G\left(x\right)^b\}dx.$$

Finally, the (s,r)th PWM of X can be obtained from an infinite linear combination of E-G moments given by

$$\rho_{s,r} = \sum_{k=0}^{\infty}\left\{\upsilon_k\,E[Y_{bk+a(r+1)}] - \upsilon_k^\star\,E[Y_{b(k+1)+a(r+1)}]\right\},$$

where $\upsilon_k^\star = d\,[bk + a(r+1)]\,\upsilon_k/[b(k+1) + a(r+1)]$.

7. Maximum-likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. The maxi-mum-likelihood estimators (MLEs) enjoy desirable properties and can be used to obtain confidence intervals for the model parameters. The normal approximation for these estimators in large samples can be easily handled either analytically or numerically. Here, we consider the estimation of the unknown parameters of the new family from complete samples only by maximum likelihood.

Let $X_1, \ldots, X_n$ be a random sample from the GT-G family with parameters λ, a, b and $\boldsymbol{\varphi}$. Let $\Theta = (\lambda, a, b, \boldsymbol{\varphi}^\intercal)^\intercal$ be the $p \times 1$ parameter vector. To obtain the MLE of Θ, the log-likelihood function is given by

$$\ell = \ell(\Theta) = \sum_{i=1}^{n} \log g\left(x_i; \varphi\right) + (a-1) \sum_{i=1}^{n} \log G\left(x_i; \varphi\right) + \sum_{i=1}^{n} \log Q_i,$$

where $Q_i = \{a(1+\lambda) - \lambda(a+b)G(x_i; \varphi)^b\}$.

Let $Z_i = G\left(x_i; \varphi\right)^b \log G\left(x_i; \varphi\right)$ and $S_{i,k} = G\left(x_i; \varphi\right)^{b-1}\left(\partial G\left(x_i; \varphi\right)/\partial \varphi_k\right)$.

The components of the score vector, $\mathbf{U}(\Theta) = \frac{\partial \ell}{\partial \Theta} = (\frac{\partial \ell}{\partial \lambda}, \frac{\partial \ell}{\partial a}, \frac{\partial \ell}{\partial b}, \frac{\partial \ell}{\partial \varphi_k})^\intercal = (U_\lambda, U_a, U_b, U_{\varphi_k})^\intercal$, are

$$U_\lambda = \sum_{i=1}^{n} \frac{1}{Q_i}\left\{a - (a+b)\, G\left(x_i; \varphi\right)^b\right\},$$

$$U_a = \sum_{i=1}^{n} \log G\left(x_i; \varphi\right) + \sum_{i=1}^{n} \frac{1}{Q_i}\left\{(1+\lambda) - \lambda G\left(x_i; \varphi\right)^b\right\},$$

$$U_b = \sum_{i=1}^{n} \frac{-1}{Q_i}\left\{\lambda\,(a+b)\, Z_i + \lambda G\left(x_i; \varphi\right)^b\right\}$$

and

$$U_{\varphi_k} = \sum_{i=1}^{n} \frac{1}{g\left(x_i; \varphi\right)} \frac{\partial g\left(x_i; \varphi\right)}{\partial \varphi_k} - \sum_{i=1}^{n} \frac{\left\{\lambda b\,(a+b)\, S_{i,k}\right\}}{Q_i} + (a-1) \sum_{i=1}^{n} \frac{1}{G\left(x; \varphi\right)} \frac{\partial G\left(x_i; \varphi\right)}{\partial \varphi_k}.$$

Setting the non linear system of equations $U_\lambda = U_a = U_b = 0$ and $U_{\varphi_\mathbf{k}} = \mathbf{0}$ and solving them simultaneously yields the MLE $\widehat{\Theta} = (\widehat{\lambda}, \widehat{a}, \widehat{b}, \widehat{\varphi}^\intercal)^\intercal$. For doing this, it is usually more convenient to adopt non linear optimization methods such as the quasi-Newton algorithm to maximize ℓ numerically. For interval estimation of the parameters, we obtain the $p \times p$ observed information matrix $J(\Theta) = \{\frac{\partial^2 \ell}{\partial r\, \partial s}\}$ (for $r, s = \lambda, a, b, \varphi$), whose elements can be computed numerically.

The standard likelihood regularity conditions are satisfied for the GT-G family. These conditions are: (i) the support of the distribution does not depend on unknown parameters; (ii) the parameter space is open and the log-likelihood function has a global maximum in it; (iii) the third order log likelihood derivatives have finite expected values; (iv) the fourth order log likelihood derivatives exist almost everywhere, and are continuous in an open neighborhood that contains the true parameter value; (v) the expected information matrix positive definite and finite. They hold for almost all distributions which satisfy (i). So, they are not restrictive.

Under standard regularity conditions when $n \to \infty$, the distribution of $\widehat{\Theta}$ can be approximated by a multivariate normal $N_p(0, J(\widehat{\Theta})^{-1})$ distribution to obtain confidence intervals for the parameters. Here, $J(\widehat{\Theta})$ is the total observed information matrix evaluated at $\widehat{\Theta}$. The method of the re-sampling bootstrap can be used for correcting the biases of the MLEs of the model parameters. Good interval estimates may also be obtained using the bootstrap percentile method. The elements of $J(\Theta)$ are given in the Appendix. Improved MLEs can be obtained for the new family using second-order bias corrections. However, these corrected estimates depend on cumulants of log-likelihood derivatives and will be addressed in future research.

8. Applications

In this section, we provide three applications to real data to illustrate the flexibility of the GT-W, GT-LL, and GT-BrX models presented in Section 3. The goodness-of-fit statistics for these models are compared with other competitive models and the MLEs of the model parameters are determined.

8.1. The nicotine data

The first data set refers to nicotine measurements, made from several brands of cigarettes in 1998, collected by the Federal Trade Commission which is an independent agency of the US government, whose main mission is the promotion of consumer protection. The report entitled tar, nicotine, and carbon monoxide of the smoke of 1206 varieties of domestic cigarettes for the year of 1998 consists of the data sets and some information about the source of the data, smokers behavior and beliefs about nicotine, tar and carbon monoxide contents in cigarettes. The free form data set can be found at *http://pw1.netcom.com/rdavis2/smoke.html*. We compare the fit of the GT-W distribution with those of other competitive models, namely: the Kumaraswamy Weibull (Kw-W), McDonald Weibull (Mc-W), gamma Weibull (GW), transmuted modified Weibull (TMW), beta Weibull (BW), modified beta Weibull (MBW) and Weibull (W) distributions with corresponding densities (for $x > 0$):

- The Kw-W density (Cordeiro et al., 2010) given by
  $f(x) = ab\beta\alpha^{\beta}x^{\beta-1}\,\mathrm{e}^{-(\alpha x)^{\beta}}(1-\mathrm{e}^{-(\alpha x)^{\beta}})^{a-1}\{1-[1-\mathrm{e}^{-(\alpha x)^{\beta}}]^{a}\}^{b-1}$.
- The Mc-W density (Cordeiro et al., 2014) given by
  $f(x) = \frac{\beta c\alpha^{\beta}}{B(a/c,b)}\,x^{\beta-1}\,\mathrm{e}^{-(\alpha x)^{\beta}}(1-\mathrm{e}^{-(\alpha x)^{\beta}})^{a-1}\{1-(1-\mathrm{e}^{-(\alpha x)^{\beta}})^{c}\}^{b-1}$.
- The GW density (Provost et al., 2011) given by
  $f(x) = \frac{\beta\alpha^{\gamma/\beta+1}x^{\beta+\gamma-1}e^{-\alpha x^{\beta}}}{\Gamma(1+\gamma/\beta)}$.
- The TMW density (Khan and King, 2013) given by
  $f(x) = (\alpha+\gamma\beta x^{\beta-1})e^{-(\alpha x+\gamma x^{\beta})}\{1-\lambda+2\lambda e^{-(\alpha x+\gamma x^{\beta})}\}$.
- The BW density (Lee et al., 2007) given by
  $f(x) = \frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}e^{-b(\alpha x)^{\beta}}[1-e^{-(\alpha x)^{\beta}}]^{a-1}$.
- The MBW density (Khan, 2015) given by
  $f(x) = \frac{\beta\alpha^{-\beta}c^{a}}{B(a/c,b)}\,x^{\beta-1}\,\mathrm{e}^{-b(\frac{x}{\alpha})^{\beta}}(1-\mathrm{e}^{-(\frac{x}{\alpha})^{\beta}})^{a-1}$
  $\times\{1-(1-c)(1-\mathrm{e}^{-(\frac{x}{\alpha})^{\beta}})^{c}\}^{-a-b}$.

The parameters of the above densities are all positive real numbers except for the TMW distribution for which $|\lambda| \le 1$.

8.2. The cancer patient data

The second data set on the remission times (in months) of a random sample of 128 bladder cancer patients (Lee and Wang, 2003) is given by 0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39,10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 5.49, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69. For these data, we compare the fit of the GT-LL distribution with those of the transmuted complementary Weibull geometric (TCWG), transmuted Weibull Lomax (TWL), transmuted linear exponential (TLE) and exponentiated transmuted generalized Rayleigh (ETGR) models ($x > 0$ for all of them).

- The TCWG density (Afify et al., 2014) given by
  $f(x) = \frac{\alpha\beta\gamma(\gamma y)^{\beta-1}\,\mathrm{e}^{-(\gamma y)^{\beta}}[\alpha(1-\lambda)-(\alpha-\alpha\lambda-\lambda-1)\,\mathrm{e}^{-(\gamma y)^{\beta}}]}{(\alpha+(1-\alpha)\,\mathrm{e}^{-(\gamma y)^{\beta}})^{3}}.$
- The TWL density (Afify et al., 2015) given by
  $f(x) = \frac{ab\alpha}{\beta}(1+\frac{x}{\beta})^{b\alpha-1}\exp\{-a[(1+\frac{x}{\beta})^{\alpha}-1]^{b}\}$
  $\times[1-(1+\frac{x}{\beta})^{-\alpha}]^{b-1}\{1-\lambda+2\lambda\mathrm{e}^{-a\{(1+\frac{x}{\beta})^{\alpha}-1\}^{b}}\}.$
- The TLE density (Tian et al., 2014) given by
  $f(x) = (\alpha+\gamma x)[1-\mathrm{e}^{-(\alpha x+\frac{\gamma}{2}x^{2})}]\{1-\lambda+2\lambda\mathrm{e}^{-(\alpha x+\frac{\gamma}{2}x^{2})}\}.$
- The ETGR density (Afify et al., 2015) given by
  $f(x) = 2\alpha\delta\beta^{2}\,x\,\mathrm{e}^{-(\beta x)^{2}}\,(1-\mathrm{e}^{-(\beta x)^{2}})^{\alpha\delta-1}$
  $\times[1+\lambda-2\lambda(1-\mathrm{e}^{-(\beta x)^{2}})^{\alpha}]\,\{1+\lambda-\lambda(1-\mathrm{e}^{-(\beta x)^{2}})^{\alpha}\}^{\delta-1}.$

The parameters of the above densities are all positive real numbers except the parameter λ where $|\lambda| \leq 1$.

8.3. The gauge lengths

The third data set (gauge lengths of 20 mm) (Kundu and Raqab, 2009) consists of 74 observations: 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585. For these data, we compare the fit of the GT-BrX distribution with those of the exponentiated transmuted generalized Rayleigh (ETGR) (here we refer it by ET-BrX), T-BrX, BrX and Rayleigh (R) models.

In order to compare the fitted models, we consider some goodness-of-fit measures including the Akaike information criterion (AIC), consistent Akaike information criterion ($CAIC$), Hannan–Quinn information criterion ($HQIC$), Bayesian information criterion (BIC) and $-2\widehat{\ell}$, where $\widehat{\ell}$ is the maximized log-likelihood, $AIC = -2\widehat{\ell} + 2p$, $CAIC = -2\widehat{\ell} + 2pn/\left(n-p-1\right)$, $HQIC = -2\widehat{\ell} + 2p\log\left[\log(n)\right]$ and $BIC = -2\widehat{\ell} + p\,\log(n)$, p is the number of parameters and n is the sample size. Further, we adopt the Anderson-Darling (A^{*}) and Cramér-von Mises (W^{*}) statistics in order to compare the fits of the two new models with other nested and non nested models. The statistics are widely used to determine how

Table 3. The statistics $-2\widehat{\ell}$, *AIC*, *BIC*, *HQIC* and *CAIC* for the nicotine data.

	Goodness-of-fit criteria						
Model	$-2\widehat{\ell}$	*AIC*	*CAIC*	*HQIC*	*BIC*	*W**	*A**
GT-W	212.167	222.167	222.344	229.826	241.399	0.36131	1.96785
TMW	217.219	225.219	225.336	231.345	240.604	0.37111	2.07974
W	226.581	230.581	230.616	233.644	238.274	0.55744	3.20719
BW	225.173	233.173	233.29	239.30	248.559	0.49664	2.89774
Kw-W	255.044	263.044	263.162	269.171	278.43	0.99745	5.82509
Mc-W	302.714	312.714	312.89	320.372	331.946	1.61799	9.5949
GW	304.503	310.503	310.573	315.098	322.042	1.64383	9.7534
MBW	330.763	340.763	340.939	348.421	359.995	1.93343	11.52508

closely a specific cdf fits the empirical distribution of a given data set. These statistics are given by

$$A^* = \left(\frac{9}{4n^2} + \frac{3}{4n} + 1\right)\left\{n + \frac{1}{n}\sum\nolimits_{j=1}^{n}(2j-1)\log\left[z_i\left(1 - z_{n-j+1}\right)\right]\right\}$$

and

$$W^* = \left(\frac{1}{2n} + 1\right)\left\{\sum\nolimits_{j=1}^{n}\left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\},$$

Table 4. MLEs and their standard errors (in parentheses) for the nicotine data.

Model	Estimates		
GT-W	$\widehat{\alpha}$= 1.5074 (0.293)	$\widehat{\beta}$= 1.8743 (0.35)	$\widehat{\lambda}$= −0.8028 (0.082)
	$\widehat{a}$= 0.854 (0.31)	$\widehat{b}$= 2.876 (1.609)	
TMW	$\widehat{\alpha}$= 0.3255 (0.315)	$\widehat{\beta}$= 2.5962 (0.244)	$\widehat{\gamma}$= 1.2691 (0.22)
	$\widehat{\lambda}$= −0.7616 (0.242)		
W	$\widehat{\alpha}$= 1.0477 (0.022)	$\widehat{\beta}$= 2.7208 (0.114)	
Kw-W	$\widehat{\alpha}$= 2.5072 (1.191)	$\widehat{\beta}$= 0.4839 (0.076)	$\widehat{a}$= 11.8142 (3.707)
	$\widehat{b}$= 18.7953 (6.212)		
Mc-W	$\widehat{\alpha}$= 1.3078 (0.596)	$\widehat{\beta}$= 0.5317 (0.079)	$\widehat{a}$= 16.858 (4.219)
	$\widehat{b}$= 10.1043 (3.995)	$\widehat{c}$= 1.1644 (0.793)	
GW	$\widehat{\alpha}$= 25.276088 (6.912)	$\widehat{\beta}$= 0.441494 (0.066)	$\widehat{\gamma}$= 9.708485 (1.337)
BW	$\widehat{\alpha}$= 0.6686 (0.578)	$\widehat{\beta}$= 3.1645 (0.426)	$\widehat{a}$= 0.7784 (0.163)
	$\widehat{b}$= 3.0922 (8.174)		
MBW	$\widehat{\alpha}$= 2.5098 (0.838)	$\widehat{\beta}$= 0.6265 (0.062)	$\widehat{a}$= 20.6338 (4.202)
	$\widehat{b}$= 9.4085 (3.163)	$\widehat{c}$= 3.6447 (0.4)	

Table 5. The statistics $-2\widehat{\ell}$, *AIC*,*BIC*, *HQIC*, and *CAIC* for the cancer data.

	Goodness-of-fit criteria						
Model	$-2\widehat{\ell}$	*AIC*	*CAIC*	*HQIC*	*BIC*	W^*	A^*
GT-LL	819.398	829.398	829.89	835.192	843.658	0.01641	0.10632
TCWG	821.995	829.995	830.32	834.63	841.403	0.04274	0.3059
TWL	820.402	830.402	830.894	836.196	844.662	0.03377	0.22162
TLE	826.971	832.971	833.165	836.448	841.528	0.06085	0.55402
ETGR	858.35	866.35	866.675	870.985	877.758	0.39794	2.36077

respectively, $z_i = F(y_j)$, where the y_j's values are the ordered observations. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness-of-fit statistics were tabulated in Nichols and Padgett (2006).

Tables 3, 5, and 7 list the values of $-2\widehat{\ell}$, *AIC*, *CAIC*, *HQIC*, *BIC*, W^*, and A^*, whereas the MLEs and their corresponding standard errors (in parentheses) of the model parameters are given in Tables 4, 6, and 8. These numerical results are obtained using the MATH-CAD PROGRAM.

In Table 3, we compare the fits of the GT-W model with the Kw-W, Mc-W, GW, TMW, BW, MBW, and W models. We note that the GT-W model has the lowest values for the $-2\widehat{\ell}$, *AIC*, *CAIC*, *HQIC*, *BIC*, W^*, and A^* statistics (for the nicotine data) among the fitted models. So, the GT-W model could be chosen as the best model. In Table 5, we compare the fits

Table 6. MLEs and their standard errors (in parentheses) for the cancer data.

Model	Estimates		
GT-LL	$\widehat{\alpha}= 9.229$ (1.908)	$\widehat{\beta}= 2.1723$ (0.321)	$\widehat{\lambda}= -0.00015$ (0.048)
	$\widehat{a}= 0.5852$ (0.16)	$\widehat{b}= 0.0005$ (0.087)	
TCWG	$\widehat{\alpha}= 106.0694$ (124.8)	$\widehat{\beta}= 1.7115$ (0.099)	$\widehat{\lambda}= 0.2168$ (0.61)
	$\widehat{\gamma}= 0.0095$ (0.00651)		
TWL	$\widehat{\alpha}= 0.201$ (0.18)	$\widehat{\beta}= 5.495$ (5.401)	$\widehat{\lambda}= -0.0006$ (0.505)
	$\widehat{a}= 10.5705$ (21.344)	$\widehat{b}= 1.5186$ (0.297)	
TLE	$\widehat{\alpha}= 0.0612$ (0.01)	$\widehat{\lambda}= 0.8568$ (0.203)	$\widehat{\gamma}= 3.0877 \times 10^{-5}$ (6.819×10^{-4})
ETGR	$\widehat{\alpha}= 7.3762$ (5.389)	$\widehat{\lambda}= 0.118$ (0.26)	$\widehat{\beta}= 0.0473$ (3.965×10^{-3})
	$\widehat{\delta}= 0.0494$ (0.036)		

Table 7. The statistics $-2\widehat{\ell}$, *AIC*, *BIC*, *HQIC*, and *CAIC* for the gauge lengths data.

	Goodness-of-fit criteria						
Model	$-2\widehat{\ell}$	*AIC*	*CAIC*	*HQIC*	*BIC*	W^*	A^*
GT-BrX	108.055	118.055	118.937	122.65	129.575	0.10458	0.68807
ET-BrX	113.4	121.352	121.9	125.029	130.6	0.20714	1.3407
T-BrX	123.61	129.61	129.95	132.376	136.5	0.16923	1.28629
BrX	135.202	139.202	139.371	141.041	143.811	0.13403	0.86836
R	188.302	190.302	190.375	191.221	192.606	1.77111	32.95987

Table 8. MLEs and their standard errors (in parentheses) for the gauge lengths data.

Model	Estimates		
GT-BrX	$\widehat{\alpha}=3.4900$ (2.084)	$\widehat{\beta}=0.6615$ (0.12)	$\widehat{\lambda}=0.0019$ (0.048)
	$\widehat{a}=2.5190$ (1.503)	$\widehat{b}=0.0161$ (0.428)	
ET-BrX	$\widehat{\alpha}=2.1214$ (0.315)	$\widehat{\beta}=0.6985$ (0.040)	$\widehat{\lambda}=0.3201$ (0.228)
	$\widehat{\delta}=7.790$ (1.727)		
T-BrX	$\widehat{\alpha}=5.5052$ (0.776)	$\widehat{\beta}=0.6245$ (0.017)	$\widehat{\lambda}=0.3599$ (0.253)
BrX	$\widehat{\alpha}=7.784$ (1.625)	$\widehat{\beta}=0.6445$ (0.024)	
R	$\widehat{\beta}=0.3962$ (0.023)		

of the GT-LL model with the TCWG, TWL, TLE, and ETGR models. The figures in this table reveal that the GT-LL model has the lowest values for $-2\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^*, and A^* statistics (except for the $CAIC$ model) for the cancer data, among all fitted models. So, the GT-LL model can be chosen as the best model. In Table 7, we compare the fits of the GT-BrX model with the ET-BrX, T-BrX, BrX and R models. The results indicate that the GT-BrX model has the lowest values for $-2\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^*, and A^* statistics (for the gauge lengths data) among all fitted models. Therefore, the GT-BrX model can be chosen as the best model. It is quite clear from the figures in Tables 3, 5, and 7 that the GT-W, GT-LL, and GT-BrX models can provide the best fits to these data. Therefore, we prove that these new distributions can be better models than other competitive lifetime models.

9. Conclusions

The idea of generating new extended models from classic ones has been of great interest among researchers in the past decade. We present a new *generalized transmuted-G* (GT-G) family of distributions, which extends the transmuted class (Shaw and Buckley, 2007) by adding two extra shape parameters. Many well-known distributions emerge as special cases of the proposed family by taking integer parameter values. We provide some mathematical properties of the new family including explicit expansions for the ordinary and incomplete moments, mean deviations, generating function, Rényi and q-entropies, order statistics and probability weighted moments. The maximum likelihood estimation of the model parameters is investigated and the observed information matrix is determined. By means of three real data sets, we verify that special cases of the GT-G family can provide better fits than other models generated from well-known families.

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Appendix

The elements of the observed matrix $J(\Theta)$ are given as follows:

$$U_{\lambda\lambda} = \sum_{i=1}^{n} \frac{-1}{Q_i^2} \left\{a - (a+b)\, G(x_i;\varphi)^b\right\}^2,$$

$$U_{\lambda a} = \sum_{i=1}^{n} \frac{1}{Q_i} \left\{1 - G(x_i;\varphi)^b\right\} - \sum_{i=1}^{n} \frac{1}{Q_i^2} \left\{a - (a+b)\, G(x_i;\varphi)^b\right\} \times \left\{(1+\lambda) - \lambda G(x_i;\varphi)^b\right\},$$

$$U_{\lambda b} = \sum_{i=1}^{n} \frac{-1}{Q_i} \left\{(a+b)\, Z_i + G(x_i;\varphi)^b\right\} + \sum_{i=1}^{n} \frac{1}{Q_i^2} \left\{a - (a+b)\, G(x_i;\varphi)^b\right\} \times \left\{\lambda\,(a+b)\, Z_i + \lambda G(x_i;\varphi)^b\right\},$$

$$U_{\lambda\varphi_k} = b\,(a+b) \sum_{i=1}^{n} \frac{-S_{i,k}}{Q_i} + \lambda b\,(a+b) \sum_{i=1}^{n} \frac{S_{i,k}}{Q_i^2} \left\{a - (a+b)\, G(x_i;\varphi)^b\right\},$$

$$U_{aa} = \sum_{i=1}^{n} \frac{-1}{Q_i^2} \left\{(1+\lambda) - \lambda G(x_i;\varphi)^b\right\}^2,$$

$$U_{ab} = \sum_{i=1}^{n} \frac{-\lambda Z_i}{Q_i} + \sum_{i=1}^{n} \frac{(1+\lambda) - \lambda G(x_i;\varphi)^b}{Q_i^2} \left\{\lambda\,(a+b)\,Z_i + \lambda G(x_i;\varphi)^b\right\},$$

$$U_{a\varphi_k} = \sum_{i=1}^{n} \frac{1}{G(x_i;\varphi)} \frac{\partial G(x_i;\varphi)}{\partial \varphi_k} - \lambda b \sum_{i=1}^{n} S_{i,k} Q_i^{-1}$$

$$+ \lambda b\,(a+b) \sum_{i=1}^{n} \frac{S_{i,k}}{Q_i^2} \left\{(1+\lambda) - \lambda G(x_i;\varphi)^b\right\},$$

$$U_{bb} = \lambda(a+b) \sum_{i=1}^{n} \frac{-1}{Q_i} \left\{\left[\log G(x_i;\varphi)\right]^2 G(x_i;\varphi)^b\right\} - 2\lambda \sum_{i=1}^{n} \frac{Z_i}{Q_i}$$

$$- \sum_{i=1}^{n} \frac{1}{Q_i^2} \left\{\lambda\,(a+b)\,Z_i + \lambda G(x_i;\varphi)^b\right\}^2,$$

$$U_{b\varphi_k} = \lambda\,(a+b) \sum_{i=1}^{n} \frac{-1}{Q_i} \frac{\partial Z_i}{\partial \varphi_k} - \lambda b \sum_{i=1}^{n} Q_i^{-1} G(x_i;\varphi)^{b-1} \frac{\partial G(x_i;\varphi)}{\partial \varphi_k}$$

$$- \lambda^2 b\,(a+b)^2 \sum_{i=1}^{n} \frac{1}{Q_i^2} Z_i G(x_i;\varphi)^{b-1} \left[\partial G(x_i;\varphi)/\partial \varphi_k\right]$$

$$- (a+b)\,b\lambda^2 \sum_{i=1}^{n} \frac{1}{Q_i^2} G(x_i;\varphi)^{2b-1} \left[\partial G(x_i;\varphi)/\partial \varphi_k\right]$$

and

$$U_{\varphi_k\varphi_k} = \sum_{i=1}^{n} g(x;\varphi)^{-1} \frac{\partial^2 g(x_i;\varphi)}{\partial \varphi_k^2} - \sum_{i=1}^{n} \frac{\left[\partial g(x_i;\varphi)/\partial \varphi_k\right]^2}{g(x_i;\varphi)^2}$$

$$+ (a-1) \sum_{i=1}^{n} G(x;\varphi)^{-1}\, \partial^2 G(x_i;\varphi)/\partial \varphi_k^2$$

$$- \lambda b\,(a+b) \sum_{i=1}^{n} \frac{G(x_i;\varphi)^{b-1}}{Q_i} \frac{\partial^2 G(x_i;\varphi)}{\partial \varphi_k^2}$$

$$- (a-1) \sum_{i=1}^{n} G(x;\varphi)^{-2} \left[\frac{\partial G(x_i;\varphi)}{\partial \varphi_k}\right]$$

$$- \lambda b\,(a+b) \sum_{i=1}^{n} \frac{(b-1)\,G(x_i;\varphi)^{b-2}}{Q_i} \left[\frac{\partial G(x_i;\varphi)}{\partial \varphi_k}\right]$$

$$+ \sum_{i=1}^{n} \frac{1}{Q_i^2} \left[\frac{\partial G(x_i;\varphi)}{\partial \varphi_k}\right]^2 \left\{\lambda b\ (a+b)\ G(x_i;\varphi)^{b-1}\right\}^2.$$